MATHEMATICS

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AIM POINTS
MATHEMATICS
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XI^th^, XII^th^, TARGET IIT-JEE (MAIN + ADVANCE) & COMPETITIVE EXAM. FOR XII (PQRS)

MAXIMA AND MINIMA

& Their Properties

CONTENTS

THINGS TO REMEMBER

1. (i) Let f(x) be a function with domain $D \subset R$. Then, f(x) is said to attain the maximum value at a point $a \in D$, if

$f(x) \leq f(a)$ for all $x \in D$.

In such a case, a is called the point of maximum and f(a) is known as the maximum value or the greatest value or the absolute maximum value of f(x).

(ii) Let f(x) be a function with domain $D \subset R$. Then f(x) is said to attain the minimum value at a point $a \in D$, if $f(x) \geq f(a)$ for all $x \in D$.

In such a case, the point a is called the point of minimum and f(a) is known as the minimum value or the least value or the absolute minimum value of f(x).

(iii) A function f(x) is said to attain a local maximum at x = a if there exists a neighbourhood $(a - \delta, a + \delta)$ of a such that

$f(x) < f(a)$ for all $x \in (a - \delta, a + \delta), x \neq a$

or, $f(x) - f(a) < 0$ for all $x \in (a - \delta, a + \delta), x \neq a$

In such a case f(a) is called the local maximum value of f(x) at x = a.

(iv) A function f(x) is said to attain a local minimum at x = a if there exists a neighbourhood $(a - \delta, a + \delta)$ of a such that

$f(x) > f(a)$ for all $x \in (a - \delta, a + \delta), x \neq a$

or, $f(x) - f(a) > 0$ for all $x \in (a - \delta, a + \delta), x \neq a$

The value of the function at x = a i.e., f(a) is called the local minimum value of f(x) at x = a.

The points at which a function attains either the local maximum values or local minimum values are known as the extreme points or turning points and both local maximum and local minimum values are called the extreme values of f(x).

Thus, a function attains an extreme value at x = a if f(a) is either a local maximum value or a local minimum value. Consequently at an extreme point 'a', f(x) - f(a) keeps the same sign for all values of x in a deleted nbd of a.

2. (First derivative test for local maxima and minima) Let f(x) be a function differentiable at x = a. Then,

(a) x = a is a point of local maximum of f(x), if

(i) $f'(a) = 0$ and

(ii) $f'(x)$ changes sign from positive to negative as x passes through a i.e., $f'(x) > 0$ at every point in the left nbd$(a - \delta)$ of a and $f'(x) < 0$ at every point in the right nbd $(a, a + \delta)$ of a.

(b) x = a is a point of local minimum f(x), if

(i) $f'(a) = 0$ and

(ii) $f'(x)$ changes sign from negative to positive as x passes through a i.e., $f'(x) < 0$ at every point in the left nbd$(a - \delta)$ of a and $f'(x) > 0$ at every point in the right nbd $(a, a + \delta)$ of a.

(c) If $f'(a) = 0$, but $f'(x)$ does not changes sign, that is, $f'(a)$ has the same sign in the complete nbd of a, then a is neither a point of local maximum nor a point of local minimum.

4. (Higher order derivative test) Let f be a differential function ona interval I and let c be an interior point of I such that

(i) $f'(c) = f''(c) = f'''(c) = ... = f^{n-1}(c) = 0$, and

(ii) $f^n(c)$ exists and is non-zero.

Then,

(a) if n is even and $f^n(c) < 0 \Rightarrow x = c$ is a point of local maximum.

(b) if n is even said $f^n(c) > 0 \Rightarrow x = c$ is a point of local minimum.

(c) if n is odd $\Rightarrow x = c$ is neither a point of local maximum nor a point of local minimum.

EXERCISE-1

1. Find the maximum and minimum values of the following functions : $f(x) = x^3 + 1$ for all $x \in R$.
2. Find the maximum and minimum values, if any, without using derivatives of the following :
   (i) $f(x) = -(x - 1)^2 + 2$ on R
   (ii) $f(x) = \sin 2x + 5$ on R.
   (iii) $f(x) = -|x + 1| + 3$ on R
   (iv) $f(x) = \sin 2x + 5$ on R
3. Find all the points of local maxima and minima of the function $f(x) = x^3 - 6x^2 + 9x - 8$
4. Find the local maxima or local minima, if any, of the function

   $$f(x) = \sin x + \cos x,\ 0 < x < \frac{\pi}{2}$$

   using the first derivative test.
5. Find the points at which the function f given by $f(x) = (x - 2)^4 (x + 2)^4 (x + 1)^3$ has
   (i) local maxim (ii) local minima (iii) points of inflexion
6. Find the points of local maxima or local minima, if any, of the following functions, using the first derivative test. Also find the local maximum or local minimum values, as the case may be :
   (i) $f(x) = x^3 - 3x$
   (ii) $f(x) = x^3 - 6x^2 + 9x + 15$
   (iii) $f(x) = x\sqrt{1-x},\ x > 0$
   (iv) $f(x) = \dfrac{1}{x^2+2}$
7. Find the points of local maxima or local minima, if any, of the following functions. Find also the local maximum or local minimum values, as the case may be :
   (i) $f(x) = \sin x + \cos x$, where $0 < x < \dfrac{\pi}{2}$
   (ii) $f(x) = \sin x - \cos x$, where $0 < x < 2\pi$
   (iii) $f(x) = \sin 2x$, where $0 < x < \pi$
   (iv) $f(x) = 2\cos x + x$, where $0 < x < \pi$
   (v) $f(x) = 2\sin x - x,\ \dfrac{-\pi}{2} \le x \le \dfrac{\pi}{2}$

8. Show that the maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{1/e}$.

9. Sow that $\sin^p \theta \cos^q \theta$ attains a maximum, when $\theta = \tan^{-1}\sqrt{\frac{p}{q}}$.

10. If $f(x) = a \log |x| + bx^2 + x$ has extreme values at $x = -1$ and at $x = 2$, then find a and b.

11. Find the points of local maxima or local minima and corresponding local maximum and local minimum values of each of the following functions. Also, find the points of inflection, if any :
    (i) $f(x) = x^4 - 62x^2 + 120x + 9$
    (ii) $f(x) = x^3 - 6x^2 + 9x + 15$
    (iii) $f(x) = (x - 1)(x + 2)^2$
    (iv) $f(x) = \frac{2}{x} + \frac{2}{x^2}, x > 0$
    (v) $f(x) = x e^x$
    (vi) $f(x) = \frac{x}{2} + \frac{2}{x}, x > 0$
    (vii) $f(x) = (x + 1)(x + 2)^{1/3} = x \geq -2$
    (viii) $f(x) = x\sqrt{32 - x^2}, -5 \leq x \leq 5$

12. The function $y = a \log x + bx^2 + x$ has extreme values at $x = 1$ and $x = 2$. Find a and b.

13. Show that $\frac{\log x}{x}$ has a maximum value at $x = e$.

14. Find the maximum and minimum values of $f(x) = 2x^3 - 24x + 107$ in the interval [1, 3].

15. Find the maximum and minimum values of $f(x) = x + \sin 2x$ in the interval $[0, 2\pi]$.

16. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals
    (i) $f(x) \left(\frac{1}{2}, x\right)^2 + x^3$, in [-2, 2.5]
    (ii) $f(x) = \sin x + \cos x$ in $[0, \pi]$

17. It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value on the interval [0, 2]. Find the value of a.

18. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals :
    (i) $f(x) = 4x - \frac{x^2}{2}$ in [-2, 4.5]
    (ii) $f(x) = (x -)^2 + 3$ in [-3, 1]

(iii) $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$ in $[0, 3]$

(iv) $f(x) = (x - 2)\sqrt{x-1}$ in $[1, 9]$

19. Find the maximum value of $2x^3 - 24x + 107$ in the interval $[1, 3]$. Find the maximum value of the same function in $[-3, -1]$.
20. Find the absolute maximum and minimum values of the function f given by
    $f(x) = \cos^2 x + \sin x,\ x \in [0, \pi]$
21. Find absolute maximum and minimum values of a function f given by
    $f(x) = 12x^{4/3} - 6x^{1/3},\ x \in [-1, 1]$
22. Find the absolute maximum and minimum values of a function f given by
    $f(x) = 2x^3 - 15x^2 + 36x + 1$ on the interval $[1, 5]$
23. Find two numbers whose sum is 24 and whose product is as large as possible.
24. Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.
25. Find two positive numbers x and y such that their sum is 35 and the product $x^2 y^5$ is maximum.
26. Show that of all the rectangles of given area, the square has the smallest perimeter.
27. Show that of all the rectangles inscribed in a given circle, the square has the iaximum area.
28. Show that the rectangle of maximum perimeter which can be inscribed in a circle of radius a is a square of side $\sqrt{2}a$.
29. If the sum of the lengths of the hypotenues and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$.
30. Prove that the area of right-angled triangle of given hypoteneus is maximum when the triangle is isosceles.
31. Show that the triangle of maximum area that can be inscribed in a given circle is an equilateral triangle.
32. A fiture consists of a semi-circle with a rectangle on its diameter. Given the perimeter of the figure, find its dimensions in order that the area may be maximum.
33. Find the volume of the largest cylinder that can be inscribed in a sphere of radius r cm.
34. Show that the height of the closed cylinder of given surface and maximum volume, is equal to the diameter of its base.
35. Show that the height of a cylinder, which is open at the top, having a given surface area greatest volume, is equal to the radius of its base.
36. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius is a $\frac{2a}{\sqrt{3}}$.
37. Show that the semi-vertical angle of a cone of maximum volume and given slant height is $\tan^{-1}\sqrt{2}$.
38. Show that the semi-vertical angle of a right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$

39. Show that the volume of the largest cone that can be inscribed in a sphere of raius R is $\frac{8}{27}$ of the volume of the sphere.

40. Prove that the radius of the right circular cylinder of greatest curved surface which can be inscribed in a given cone is half of that of the cone.

41. Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle α is $\frac{4}{27}\pi h^3 \tan^2\alpha$. Also, show that height of the cylinder is $\frac{h}{3}$.

42. An open box with a square base is to be made out of a given quantity of card board of area c_2 square units. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units.

43. The combined resistance R of two resistors R_1 and R_2 ($R_1, R_2 > 0$) is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

If $R_1 + R_2 = C$ (a constant),show that the maximum resistance R is obtained by choosing $R_1 = R_2$.

44. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of the material will be least when depth of the tank is half of its width.

45. Find the area of the greatest isosceles triangle that can be inscribed in a given ellipse having its vertex coincident with one end of the major axis.

46. A point on the hypotenuse of a right triangle is at distance a and b from the sides of the triangle. Show that the minimum length of the hypotenuse is $(a^{2/3} + b^{2/3})^{3/2}$.

47. Find the shortest distance of the point (0, c) from the parabola $y = x^2$, where $0 \le c \le 5$.

48. Let AP and BQ be two vertical poles at points A and B respectively. If AP = 16 m, BQ = 22 m and AB = 20 m, then find the distance of a point R on AB from the point A such that $RP^2 + RQ^2$ is minimum.

49. Of all the closed cylindrical cans (right circular), which enclose a given volume of 100 cm^3, which has the minimum surface area ?

50. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off squares from each corners and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum possible ?

51. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$.

53. A wire of length 20 m is to be cut into two pieces. One of the pieces will be bent into shape of a square and the other into shape of an equilateral triangle. Where the wire should be cut so that the sum of the areas of the square and triangle is minimum.

54. A rectangle is inscribed in a semi-circle of radius r with one of its sides ondiameter of semi-circle. Find the dimensions of the rectangle so that its area is maximum. Find also the area.

55. Given the sum of the perimeters of a square and a circle, show that the sum of their areas is least when one side of the square is equal to diameter of the circle.

56. A window in the form of a rectangle is surmounted by a semi-circular opening. The total perimeter of the window is 10 m. Find the dimensions of the rectangular part of the window to admit maximum light through the whole opening.

57. Find the dimensions of the rectangle of perimeter 36cm which will sweep out a volume as large as possible when revolved about one of its sides.

58. Find the largest possible area of a right angled triangle whose hypotenuse is 5 cm long.

59. Two sides of a triangle have lengths 'a' and 'b' and the angle between them is θ. What value of θ will maximize the area of the triangle ? Find the maximum area of the triangle also.

60. Manufacturer can sell x items at a price of rupees $\left(5-\frac{x}{100}\right)$ each. The cost price is Rs. $\left(\frac{x}{5}+500\right)$. Find the number of items he should sell to earn maximum profit.

61. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs Rs. 70 per square metre for the base and Rs. 45 per square metre for sides, what is the cost of least expensive tank ?

EXERCISE-2

1. Let $f(x) = x^3 + 3x^2 - 9x + 2$. Then, f(x) has
   (a) a maximum at x = 1
   (b) a minimum at x = 1
   (c) neither a maximum nor a minimum at x = -3
   (d) none of these

2. The sum of two non-zero numbers is 8, the minimum value of the sum of their reciprocals is
   (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{8}$ (d) none of these

3. The maximum value of $f(x) = \frac{x}{4-x+x^2}$ on [-1, 1] is
   (a) $-\frac{1}{4}$ (b) $-\frac{1}{3}$ (c) $\frac{1}{6}$ (d) $\frac{1}{5}$

4. $f(x) = \sin + \sqrt{3}\cos x$ is maximum when x =
   (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) 0

5. If $f(x) = x + \frac{1}{x}$, x > 0, then its greatest value is
   (a) -2 (b) 0 (c) 3 (d) none of these

6. The function $f(x) = 2x^3 - 15x^2 + 36x + 4$ is maximum at x =
   (a) 3 (b) 0 (c) 4 (d) 2